Homework 6
MTH 829 Complex Analysis

Joshua Ruiter

February 13, 2018

Lemma 0.1 (for Exercise VIII.1.1)**.** *Let $b \in \mathbb{C} \setminus \{0\}$. Then $\frac{1}{z-b}$ is represented by the series $-\sum_{n=0}^{\infty} b^{-n-1} z^n$ in the disk $|z| < |b|$.*

Proof. First, note that we can rewrite it as

$$\frac{1}{z-b} = -\frac{1}{b}\left(\frac{1}{1-\frac{z}{b}}\right)$$

Then we know that $\frac{1}{1-\frac{z}{b}}$ is represented by $\sum_{n=0}^{\infty} \left(\frac{z}{b}\right)^n$ on $|\frac{z}{b}| < 1 \iff |z| < |b|$, so the series representing $\frac{1}{z-b}$ is

$$-\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{z}{b}\right)^n = -\frac{1}{b}\sum_{n=0}^{\infty} b^{-n} z^n = -\sum_{n=0}^{\infty} b^{-n-1} z^n$$

□

Lemma 0.2 (for Exercise VIII.1.1)**.** *Let $a \in \mathbb{C} \setminus \{0\}$. Then $\frac{1}{z-a}$ is represented by the series $\sum_{n=0}^{\infty} a^n z^{-n-1}$ in the region $|a| < |z|$.*

Proof. We can rewrite it as

$$\frac{1}{z-a} = \frac{1}{z}\left(\frac{1}{1-\frac{a}{z}}\right)$$

The second factor is represented by the power series $\sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$ on $|\frac{a}{z}| < 1 \iff |a| < |z|$. Thus $\frac{1}{z-a}$ is represented by

$$\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \frac{1}{z}\sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n-1}$$

□

Proposition 0.3 (Exercise VIII.1.1)**.** *Let $a, b \in \mathbb{C}$ so that $0 < |a| < |b|$. Then the series*

$$\frac{1}{b-a}\sum_{n=-\infty}^{\infty} c_n z^n$$

represents the function $\frac{1}{(z-a)(z-b)}$ *in the annulus* $|a| < |z| < |b|$, *where*

$$c_n = \begin{cases} a^{n+1} & n < 0 \\ b^{-n-1} & n \geq 0 \end{cases}$$

Proof. We can rewrite it as

$$\frac{1}{(z-a)(z-b)} = \left(\frac{1}{b-a}\right)\left(\frac{1}{z-a}\right) + \left(\frac{1}{a-b}\right)\left(\frac{1}{z-b}\right)$$

Using the previous two lemmas, $\frac{1}{z-a}$ is represented by $\sum_{n=0}^{\infty} a^n z^{-n-1}$ on $|a| < |z|$, and $\frac{1}{z-b}$ is represented by $-\sum_{n=0}^{\infty} b^{-n-1} z^n$ on $|z| < |b|$, so on $|a| < |z| < |b|$ we can represent the above by

$$\left(\frac{1}{b-a}\sum_{n=0}^{\infty} a^n z^{-n-1}\right) - \left(\frac{1}{a-b}\sum_{n=0}^{\infty} b^{-n-1} z^n\right) = \left(\frac{1}{b-a}\sum_{n=-\infty}^{-1} a^{n+1} z^n\right) + \left(\frac{1}{b-a}\sum_{n=0}^{\infty} b^{-n-1} z^n\right)$$

If we define

$$c_n = \begin{cases} a^{n+1} & n < 0 \\ b^{-n-1} & n \geq 0 \end{cases}$$

Then we can represent $\frac{1}{(z-a)(z-b)}$ by the series

$$\frac{1}{b-a}\sum_{n=-\infty}^{\infty} c_n z^n$$

on $|a| < |z| < |b|$. □

Proposition 0.4 (Exercise VIII.2.1)**.** *The annulus of convergence for the Laurent series*

$$\sum_{n=-\infty}^{\infty} a^{n^2} z^n$$

is $0 < |z| < \infty$.

Proof. Applying the Cauchy-Hadamard theorem, the annulus of convergence is $R_1 < |z| < R_2$ where

$$R_1 = \limsup_{n\to\infty} \left|a^{(-n)^2}\right|^{1/n} = \limsup_{n\to\infty} \left|a^{n^2}\right|^{1/n} \qquad R_2 = \left(\limsup_{n\to\infty} \left|a^{n^2}\right|^{1/n}\right)^{-1}$$

Working with these expressions, we have

$$|a^{n^2}| = |a|^{n^2} \implies \left|a^{n^2}\right|^{1/n} = \left(|a|^{n^2}\right)^{1/n} = |a|^{n^2(1/n)} = |a|^n$$

Because $|a| < 1$,

$$R_1 = \limsup_{n\to\infty} |a|^n = 0$$

So $R_2 = \infty$. Thus the annulus of convergence is $0 < |z| < \infty$. □

Proposition 0.5 (Exercise VIII.4.1)**.** *Let p, q be polynomials such that* $\deg q > \deg p + 1$. *Let C be a circle whose interior contains all of the roots of q. Then*

$$\int_C \frac{p(z)}{q(z)} dz = 0$$

Proof. Let z_0 be the center of C, and let r_0 be the infimum over all $r > 0$ so that the circle $C_r = \{z : |z - z_0| = r\}$ contains all of the roots of q. Then $\frac{p(z)}{q(z)}$ is holomorphic on $\{z : |z - z_0| > r_0\}$, and

$$\int_{C_r} \frac{p(z)}{q(z)} dz = \int_C \frac{p(z)}{q(z)} dz$$

for all $r > r_0$. Since $\deg q > \deg p + 1$, we know that the limit

$$\lim_{z \to \infty} \frac{(z - z_0)^2 p(z)}{q(z)}$$

exists and is finite, which says that on $\{z : |z - z_0| > r_0\}$, for some $M > 0$ we have

$$\left| \frac{p(z)}{q(z)} \right| \leq \frac{M}{|z - z_0|^2}$$

Thus for $r > r_0$,

$$\left| \int_C \frac{p(z)}{q(z)} dz \right| = \left| \int_{C_r} \frac{p(z)}{q(z)} dz \right| \leq \int_{C_r} \left| \frac{p(z)}{q(z)} \right| dz \leq \int_{C_r} \frac{M}{|z - z_0|^2} dz = \int_{C_r} \frac{M}{r^2} dz = \frac{M}{r^2} L(C_r) = \frac{2\pi M}{r}$$

Since this holds for all $r > r_0$, we conclude that

$$\left| \int_C \frac{p(z)}{q(z)} dz \right| = 0 \implies \int_C \frac{p(z)}{q(z)} dz = 0$$

□

Proposition 0.6 (Exercise VIII.7.2a)**.** *The isolated singularities of* $f(z) = \frac{z^5}{1+z+z^2+z^3+z^4}$ *are*

$$e^{2\pi i/5} \qquad e^{4\pi i/5} \qquad e^{6\pi i/5} \qquad e^{8\pi i/5} \qquad \infty$$

Each of the above singularities is a pole.

Proof. For $z \neq 1$, we have

$$1 + z + z^2 + z^3 + z^4 = \frac{1 - z^5}{1 - z}$$

The denominator of $f(z)$ doesn't vanish for $z = 1$, so the only finite singularities of f occur where $1 - z^5 = 0$ and $z \neq 1$. Let $\lambda = e^{2\pi i/5}$ be the principal 5th root of unity. Then the roots of the denominator of f are $\lambda, \lambda^2, \lambda^3, \lambda^4$, so these are the finite singularities of f. Now we check if ∞ is a singularity. Because

$$f\left(\frac{1}{z}\right) = \frac{1}{z + z^2 + z^3 + z^4 + z^5}$$

has a singularity at the origin, f has a singularity at ∞. Furthremore,

$$\lim_{z \to 0} \left| f\left(\frac{1}{z}\right) \right| = \lim_{z \to 0} \left| \frac{1}{z + z^2 + z^3 + z^4 + z^5} \right| = \infty$$

so by VIII.9 (Criterion for a Pole), ∞ is a pole of f. Applying this criterion to the other singularities also tells us that they are poles. For each of $\lambda, \lambda^2, \lambda^3, \lambda^4$, we can write f near the singularity as

$$f(z) = \frac{z^5(1-z)}{1-z^5}$$

thus

$$\lim_{z \to \lambda} |f(z)| = \lim_{z \to \lambda} \left| \frac{z^5(1-z)}{1-z^5} \right| = \infty$$

since the numerator is bounded and the denominator goes to zero. □

Proposition 0.7 (Exercise VIII.7.2b)**.** *The isolated singularities of $f(z) = \frac{1}{\sin^2 z}$ are πk for $k \in \mathbb{Z}$, and each singularity is a pole.*

Proof. The function f is holomorphic except when the denominator vanishes or at infinity, so the only isolated singularities are where $\sin z = 0$ or infinity. The zeroes of $\sin z$ are πk for $k \in \mathbb{Z}$. Since f has a singularity on every neighborhood of ∞, there is no neighborhood of ∞ on which f is holomorphic, so f does not have an isolated singularity at infinity.

We claim that all of these singularities are poles. We have

$$\lim_{z \to \pi k} |f(z)| = \lim_{z \to \pi k} \left| \frac{1}{\sin^2 z} \right| = \infty$$

so by the result in VIII.9, each singularity is a pole. □

Proposition 0.8 (Exercise VIII.7.2c)**.** *Let $f(z) = \sin\left(\frac{1}{z}\right)$. The isolated singularities of f are at zero and infinity. Infinity is a removable singularity, and zero is an essential singularity.*

Proof. The function f is well defined and holomorphic for $z \in \mathbb{C} \setminus \{0\}$, so the only possible isolated singularities are $0, \infty$. Infinity is a removable singularity because

$$\sin\left(\frac{1}{1/z}\right)$$

has a removable singularity at the origin. The Laurent series for $\sin\left(\frac{1}{z}\right)$ centered at zero is

$$\sin\left(\frac{1}{z}\right) = z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \dots$$

which has an unbounded principal part, so zero is an essential singularity. □

Proposition 0.9 (Exercise VIII.7.3)**.** *A rational function has no essential singularities.*

Proof. The zero function has no singularities, so suppose that $f(x) = \frac{p(x)}{q(x)}$ is a rational function where p, q are nonzero rational functions. We can write p and q as products of linear factors,

$$f(z) = \frac{A(z-a_1)\dots(z-a_n)}{B(z-b_1)\dots(z-b_k)}$$

We know that f is holomorphic everywhere it is defined, which is everywhere except $b_0, \dots, b_k$ and ∞, so these are the only possible isolated singularities. If any a_i is equal to some b_j, the "same" rational function after cancelling has no more singularities and the funtion before cancelling may have a removable singularity at the cancelled root, but not an essential singularity. So we assume that all possible cancellations are made, leaving us with a rational funciton with no removable singularities.

Now that $a_j \neq b_i$ for any i, j, we can factor out all copies of any given root, and take the limit as $z \to b_i$. For example, considering b_1,

$$\begin{aligned}\lim_{z\to b_1} |f(z)| = \lim_{z\to b_1} \left| \frac{A(z-a_1)\dots(z-a_n)}{B(z-b_1)\dots(z-b_k)} \right| &= \lim_{z\to b_1} \left| \left(\frac{1}{z-b_1} \right)^m \frac{A(b_i-a_1)\dots(b_i-a_n)}{B(b_i-b_2)\dots(b_i-b_k)} \right| \\ = \left| \frac{A(b_i-a_1)\dots(b_i-a_n)}{B(b_i-b_2)\dots(b_i-b_k)} \right| \lim_{z\to b_1} \left| \frac{1}{z-b_1} \right|^m = \infty\end{aligned}$$

so the singularity at b_1 is a pole. We can do the same factoring trick for any b_i, so each b_i is either a pole or a removable singularity.

Now consider the possible singularity at ∞. If $\deg p < \deg q$, then

$$\lim_{z\to\infty} f(z) = 0$$

which says that f is bounded in a punctured neighborhood of ∞, and hence ∞ is a removable singularity. Similarlty, if $\deg p = \deg q$, then

$$\lim_{z\to\infty} f(z) = \frac{A}{B}$$

where A, B are the leading coefficients of p, q respectively. This also implies that f is bounded in a punctured neighborhood of ∞, so it is a removable singularity. If $\deg p > \deg q$, then

$$\lim_{z\to\infty} f(z) = \infty$$

which implies that ∞ is a pole. Thus f has no essential singularities. □

Lemma 0.10 (for Exercise VIII.7.4)**.** *Let f, g be nonzero polynomials over a field k. Write g as a product of powers of distinct irreducible polynomials,*

$$g = \prod_{i=1}^{k} p_i^{n_i}$$

Then there are polynomials b and a_{ij} with $\deg a_{ij} < \deg p_i$ such that

$$\frac{f}{g} = b + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{p_i^j}$$

If $\deg f < \deg g$, then $b = 0$. In particular, if $k = \mathbb{C}$, then each p_i has degree one, so a_{ij} are complex constants.

Proof. As a base case, suppose $\deg g = 1$. Then by the Euclidean division algorithm, there exist polynomials q, r so that $f = gq + r$ and $\deg r < \deg g = 1$, so r is a constant. Thus

$$\frac{f}{g} = q + \frac{r}{g}$$

is of the required form. Note that if $\deg f < \deg g$, then $q = 0$.

As an inductive hypothesis, suppose that the result holds for $\deg g = m$, and let $\deg g = m + 1$. Then we can write g as a product of coprime polynomials P, Q each with degree strictly less than g, that is, degree less than or equal to m. By Bezout's Identity, there exist polynomials C, D such that

$$CP + DQ = 1$$

Then

$$\frac{1}{g} = \frac{CP + DQ}{PQ} = \frac{C}{Q} + \frac{D}{P} \implies \frac{f}{g} = \frac{fC}{Q} + \frac{fD}{P}$$

By induction hypothesis, $\frac{fC}{Q}$ and $\frac{fD}{P}$ can be written in the desired form, so their sum, $\frac{f}{g}$ can be written in the desired form. □

Proposition 0.11 (Exercise VIII.7.4)**.** *Let f be a nonconstant rational function with poles $z_1, \ldots, z_k \in \overline{\mathbb{C}}$. Then f can be written as $f = f_1 + \ldots + f_k$ where each f_i is a rational function whose only pole is z_i.*

Proof. Write f as $f(z) = \frac{p(z)}{q(z)}$. First we consider the case where all the poles are finite. Then $\deg p < \deg q$, since ∞ is not a pole. Furthermore, $z_1, \ldots z_n$ are roots of q, so

$$q(z) = (z - z_1)^{n_1} \ldots (z - z_k)^{n_k}$$

where k_1 are positive integers. Then by the previous lemma, we can write f as

$$f = b + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{(z - z_i)^j} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{(z - z_i)^j}$$

where $b = 0$ since $\deg p < \deg q$. We define

$$f_i = \sum_{j=1}^{n_i} \frac{a_{ij}}{(z - z_i)^j}$$

and we see that z_i is the only pole of f_i. We can then find a lowest common denominator and write f_i as a rational function, and we see that f has been written in the claimed form.

Now suppose that f has a pole at ∞. Then $f(z) = \frac{p(z)}{q(z)}$ where $\deg p \geq \deg q$, since if $\deg p < \deg q$, then ∞ is a removable singularity. Without loss of generality, assume $z_k = \infty$ and the other poles are finite. Then they are roots of q, so

$$q(z) = (z - z_1)^{n_1} \ldots (z - z_{k-1})^{n_{k-1}}$$

By the previous lemma, we can write f as

$$f = b + \sum_{i=1}^{k-1} \sum_{j=1}^{n_i} \frac{a_{ij}}{(z - z_i)^j}$$

As above, for $i = 1, \dots, k-1$, define

$$f_i = \sum_{j=1}^{n_i} \frac{a_{ij}}{(z - z_i)^j}$$

And define $f_k = b$. Then ∞ is a pole of the polynomial b, so we have written f in the desired form. □

Proposition 0.12 (Exercise VIII.7.5)**.** *Let F be a finite subset of $\mathbb{C}$ and let f be holomorphic on $\mathbb{C} \setminus F$, such that f has no essential singularities. Then f is a rational function.*

Proof. Let $F = \{\alpha_1, \dots, \alpha_n\}$. Since each α_i is either a pole or removable singularity of f, there exists $m_i \in \mathbb{Z}$ so that the limit

$$\lim_{z \to \alpha_i} (z - \alpha_i)^{m_i} f(z)$$

exists and is finite. Then define

$$g(z) = \prod_{k=1}^{n} (z - \alpha_i)^{m_i} \qquad\qquad h(z) = \begin{cases} f(z)g(z) & z \notin F \\ \lim_{w \to z} f(w)g(w) & z \in F \end{cases}$$

As already noted, the limit in the definition of h always exists and is finite, since $(z - \alpha_i)^{m_i}$ divides $g(z)$. Thus h is entire.

Since f has no essential singularities, in particular, f does not have an essential singularity at infinity. If ∞ is a removable singularity, then

$$\lim_{z \to \infty} f(z)$$

exists and is finite. If ∞ is a pole of order m, then

$$\lim_{z \to \infty} \frac{f(z)}{z^m}$$

exists and is finite. Either way, there exists $m \in \mathbb{Z}$ and $R_f > 0$ such that $\frac{f(z)}{z^m}$ is bounded on $|z| > R_f$. Similarly, since g is a polynomial, there exists $k \in \mathbb{Z}$ and $R_g > 0$ so that $\frac{g(z)}{z^k}$ is bounded on $|z| > R_g$. Thus,

$$\frac{h(z)}{z^{m+n}} = \frac{f(z)g(z)}{z^m z^n}$$

is bounded on $|z| > \max(R_f, R_g)$. Then by the homework exercise VII.11.1, h is a polynomial. Thus

$$f(z) = \frac{g(z)}{h(z)}$$

on $\mathbb{C} \setminus F$, which is a rational function as g, h are polynomials. □

Proposition 0.13 (Exercise VIII.7.6)**.** *Let f, g be holomorphic functions both with a pole of order m at z_0. Then*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}$$

Proof. We can represent both f and g locally by Laurent series centered at z_0 on punctured disks. Choose a punctured disk small enough to contain both local representation, so on a suitable punctured disk we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \qquad g(z) = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$$

Since both f, g have a pole of order m at z_0, $a_n = b_n = 0$ for $n < -m$ and $a_{-m}, b_{-m} \neq 0$. Thus

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{(z-z_0)^m f(z)}{(z-z_0)^m g(z)} = \lim_{z\to z_0} \frac{\sum_{n=-m}^{\infty} a_n(z-z_0)^{n+m}}{\sum_{n=-m}^{\infty} b_n(z-z_0)^{n+m}}$$

For the expression on the right, both the numerator and denominator have a nonzero constant term plus something divisible by $(z-z_0)$, so the limit is the ratio of the constant terms, $\frac{a_{-m}}{b_{-m}}$. Differentiating the Laurent series for f and g gives

$$f'(z) = \sum_{n=-m}^{\infty} na_n(z-z_0)^{n-1} \qquad g'(z) = \sum_{n=-m}^{\infty} nb_n(z-z_0)^{n-1}$$

Thus

$$\begin{aligned}\lim_{z\to z_0} \frac{f'(z)}{g'(z)} &= \lim_{z\to z_0} \frac{(z-z_0)^{m+1} f'(z)}{(z-z_0)^{m+1} g'(z)} = \lim_{z\to z_0} \frac{(z-z_0)^{m+1}\sum_{n=-m}^{\infty} na_n(z-z_0)^{n-1}}{(z-z_0)^{m+1}\sum_{n=-m}^{\infty} nb_n(z-z_0)^{n-1}} \\ &= \lim_{z\to z_0} \frac{\sum_{n=-m}^{\infty} na_n(z-z_0)^{n+m}}{\sum_{n=-m}^{\infty} nb_n(z-z_0)^{n+m}}\end{aligned}$$

As before, both numerator and denominator have a nonzero constant term plus something divisible by $(z-z_0)$, so the limit is the ratio of the constant terms, $\frac{-ma_{-m}}{-mb_{-m}} = \frac{a_{-m}}{b_{-m}}$. Thus

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}$$

□

Proposition 0.14 (Exercise VIII.8.2). *Let f be holomorphic with an isolated singularity at $z_0 \in \mathbb{C}$. Suppose that there exist $M, \epsilon > 0$ and a positive integer m so that*

$$0 < |z-z_0| < \epsilon \implies |f(z)| \leq M|z-z_0|^{-m}$$

Then z_0 is either a removable singularity of f or a pole of order at most m.

Proof. We can represent f by a Laurent series on some disk centered at z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

We can assume that this disk lies inside the punctured disk $0 < |z-z_0| < \epsilon$, by shrinking it if necessary. On this punctured disk, by hypothesis we have

$$|(z-z_0)^m f(z)| \leq M$$

so by the result in VIII.8 (page 105 of Sarason), z_0 is a removable singularity of $(z - z_0)^m f(z)$, which is represented by the Laurent series

$$(z - z_0)^m f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^{n+m}$$

on a punctured disk centered at z_0. Since z_0 is a removable singularity, $a_n = 0$ for $n < -m$. Applying this knowledge to the Laurent series for f, we see that f must have a removable singularity or pole of order at most m at z_0. □

Lemma 0.15 (for Exercise VIII.12.1)**.** *Let f be holomorphic on an open set containing z_0 and let f have a simple pole at z_0. Then*

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z)$$

Proof. Since f has a simple pole at z_0, there is a small punctured disk on which f has a Laurent series

$$f(z) = \sum_{n=-1}^{\infty} a_n (z - z_0)^n$$

Then on that punctured disk we can represent $(z - z_0) f(z)$ by

$$(z - z_0) f(z) = \sum_{n=-1}^{\infty} a_n (z - z_0)^{n+1}$$

which is holomorphic, since it is a power series. Thus

$$\begin{aligned}
\lim_{z \to z_0} (z - z_0) f(z) &= \lim_{z \to z_0} \sum_{n=-1}^{\infty} a_n (z - z_0)^{n+1} \\
&= \left(\lim_{z \to z_0} a_{-1} \right) + \left(\lim_{z \to z_0} (z - z_0) \sum_{n=0}^{\infty} a_n (z - z_0)^n \right) \\
&= a_{-1} \\
&= \operatorname{res}_{z_0} f
\end{aligned}$$

□

Proposition 0.16 (Exercise VIII.12.1)**.** *Let g, h be holomorphic in an open set containing z_0, and suppose that h has a simple zero at z_0. Then*

$$\operatorname{res}_{z_0} \frac{g}{h} = \frac{g(z_0)}{h'(z_0)}$$

Proof. Because h has a simple zero at z_0, we can write h as $h(z) = (z - z_0) f(z)$ where f is a nonvanishing holomorphic function on the same set as h, and $f(z_0) = h'(z_0) \neq 0$. If

$g(z_0) = 0$, then g can be written as $g(z) = (z - z_0)^m t(z)$ for some holomorphic function t, where m is the order of the zero at z_0. Then

$$\frac{g}{h} = \frac{(z - z_0)^m g(z)}{(z - z_0)h(z)}$$

has a removable singularity at z_0, so $\text{res}_{z_0} \frac{g}{h} = 0$ which is equal to $\frac{g(z_0)}{h'(z_0)} = \frac{0}{h'(z_0)} = 0$ and we have the desired equality.

Now we can assume that $g(z_0) \neq 0$. Since h has a simple zero at z_0, $\frac{g}{h}$ has a simple pole at z_0 because

$$\lim_{z \to z_0} (z - z_0)\frac{g(z)}{h(z)} = \lim_{z \to z_0} (z - z_0)\frac{g(z)}{(z - z_0)f(z)} = \lim_{z \to z_0} \frac{g(z)}{f(z)} = \frac{g(z_0)}{f(z_0)} \neq 0$$

as $g(z_0) \neq 0$. Thus by the previous lemma,

$$\text{res}_{z_0} \frac{g}{h} = \lim_{z \to z_0} (z - z_0)\frac{g(z)}{h(z)} = \frac{g(z_0)}{f(z_0)} = \frac{g(z_0)}{h'(z_0)}$$

□

Proposition 0.17 (Exercise VIII.12.1a). *Let p, q be positive integers, and define*

$$f(z) = \frac{z^p}{1 - z^q}$$

Let λ be the principal qth root of unity. Then the finite isolated singularities of f are $1, \lambda, \lambda^2, \ldots, \lambda^{q-1}$, and the residues at λ^k is given by

$$\text{res}_{\lambda^k} f = \frac{\lambda^{kp}}{-\prod_{n \neq k}^{q}(\lambda^k - \lambda^n)}$$

Proof. Let λ be the principal qth root of unity. Then we can write f as

$$f(z) = \frac{z^p}{-\prod_{n=1}^{q}(z - \lambda^n)}$$

Thus the isolated singularities of f are the roots of the denominator, which are the qth roots of unity. Each is a simple pole, since

$$\lim_{z \to \lambda^k} (z - \lambda^k)f(z) = \lim_{z \to \lambda^k} (z - \lambda^k)\frac{z^p}{-\prod_{n=1}^{q}(z - \lambda^n)} = \lim_{z \to \lambda^k} \frac{\lambda^{kp}}{-\prod_{n \neq k}^{q}(\lambda^k - \lambda^n)}$$

is finite. Thus by a lemma above (Lemma 0.15),

$$\text{res}_{\lambda^k} f = \lim_{z \to \lambda^k} (z - \lambda^k)f(z) = \frac{\lambda^{kp}}{-\prod_{n \neq k}^{q}(\lambda^k - \lambda^n)}$$

□

Lemma 0.18 (for Exercise VIII.12.2c)**.** *If f has a pole of order k at z_0, then*

$$\operatorname{res}_{z_0} f = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z)$$

Proof. We can represent f by a Laurent series on a punctured disk centered at z_0 by

$$f(z) = \sum_{n=-k}^{\infty} a_n (z-z_0)^n$$

Multiplying through by $(z-z_0)^k$, we get

$$(z-z_0)^k f(z) = a_{-k} + \ldots + a_{-1}(z-z_0)^{k-1} + a_0(z-z_0)^k + \ldots$$

After differentiating $k-1$ times, we get

$$\frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z) = (k-1)! a_{-1} + k! a_0 (z-z_0) + \ldots$$

Then taking the limit as $z \to z_0$, we get

$$\lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z) = (k-1)! a_{-1} = (k-1)! \operatorname{res}_{z_0} f$$

Then dividing by $(k-1)!$ gives the desired equality. □

Proposition 0.19 (Exercise VIII.12.2b)**.** *Define*

$$f(z) = \frac{z^5}{(z^2-1)^2}$$

Then the isolated singularities of f are ± 1, and the residues are

$$\operatorname{res}_1 f = \operatorname{res}_{-1} f = 1$$

Proof. We can rewrite f as

$$f(z) = \frac{z^5}{(z-1)^2(z+1)^2}$$

from which we can immediately read off that the only finite isolated singularities are ± 1, and we can see that they are both poles of order 2. Then we compute

$$\operatorname{res}_1 f = \frac{1}{(2-1)!} \lim_{z \to 1} \left(\frac{\partial}{\partial z} \left((z-1)^2 f(z) \right) \right) = \lim_{z \to 1} \frac{\partial}{\partial z} \left(\frac{z^5}{(z+1)^2} \right) = \lim_{z \to 1} \frac{z^4(3z+5)}{(z+1)^3} = 1$$

$$\operatorname{res}_{-1} f = \frac{1}{(2-1)!} \lim_{z \to -1} \left(\frac{\partial}{\partial z} \left((z+1)^2 f(z) \right) \right) = \lim_{z \to -1} \frac{\partial}{\partial z} \left(\frac{z^5}{(z-1)^2} \right) = \lim_{z \to -1} \frac{z^4(3z-5)}{(z-1)^3} = 1$$

□

Proposition 0.20 (Exercise VIII.12.2c). *Define*

$$f(z) = \frac{\cos z}{1 + z + z^2}$$

Then the finite singularites of f are $e^{2\pi i/3}, e^{4\pi i/3}$, and the resides at these singularities are

$$\operatorname{res} e^{2\pi i/3} f = \frac{\cos e^{2\pi i/3}}{i\sqrt{3}} \qquad \operatorname{res} e^{4\pi i/3} f = \frac{\cos e^{4\pi i/3}}{-i\sqrt{3}}$$

Proof. We can factor the denominator as $1 + z + z^2 = \left(z - e^{2\pi i/3}\right)\left(z - e^{4\pi i/3}\right)$, so the finite singularities of f are at $e^{2\pi i/3}, e^{4\pi i/3}$. These are both simple poles, since the numerator does not vanish off of the real line. Then using our formula,

$$\operatorname{res}_{e^{2\pi i/3}} f = \lim_{z \to e^{2\pi i/3}} \left(z - e^{2\pi i/3}\right) \frac{\cos z}{\left(z - e^{2\pi i/3}\right)\left(z - e^{4\pi i/3}\right)} = \frac{\cos e^{2\pi i/3}}{e^{2\pi i/3} - e^{4\pi i/3}} = \frac{\cos e^{2\pi i/3}}{i\sqrt{3}}$$
$$\operatorname{res}_{e^{4\pi i/3}} f = \lim_{z \to e^{4\pi i/3}} \left(z - e^{4\pi i/3}\right) \frac{\cos z}{\left(z - e^{2\pi i/3}\right)\left(z - e^{4\pi i/3}\right)} = \frac{\cos e^{4\pi i/3}}{e^{4\pi i/3} - e^{2\pi i/3}} = \frac{\cos e^{4\pi i/3}}{-i\sqrt{3}}$$

□

Proposition 0.21 (Exercise VIII.12.2d). *Define $f(z) = \frac{1}{\sin z}$. The finite singularities of f occur at $n\pi$ for $n \in \mathbb{Z}$, and the residues are*

$$\operatorname{res}_{n\pi} f = \cos n\pi = \begin{cases} -1 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases}$$

Proof. The singularities of f occur when the denominator vanishes, which occurs at $z = n\pi$ for $n \in \mathbb{Z}$. A zero at $n\pi$ of $\sin z$ is a simple zero because $\frac{\partial}{\partial z} \sin z = \cos z$ and $\cos(n\pi) = 0$. Thus the poles of f at $n\pi$ are simple poles. Thus applying the formula for the residue at a pole, and using the complex L'Hopital's rule,

$$\operatorname{res}_{n\pi} \sin z = \lim_{z \to n\pi} \frac{z - n\pi}{\sin z} = \lim_{z \to n\pi} \frac{1}{\cos z} = \cos n\pi = \begin{cases} -1 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases}$$

□

Proposition 0.22 (Exercise VIII.12.3). *Let f be holomorphic in an open set containing z_0, and suppose that f has a zero of order m at z_0. Then*

$$\operatorname{res}_{z_0} \frac{f'}{f} = m$$

Proof. We can write f as $f(z) = (z - z_0)^m g(z)$ where g is holomorphic and $g(z_0) \neq 0$. Then $f'(z) = m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z)$, so

$$\frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z)}{(z - z_0)^m g(z)} = \frac{mg(z) + (z - z_0) g'(z)}{(z - z_0) g(z)}$$

Since $g(z_0) \neq 0$, the denominator of this final quotient has a simple pole at z_0, so by Exercise VIII.12.1,

$$\begin{aligned}
\operatorname{res}_{z_0} \frac{f'}{f} &= \operatorname{res}_{z_0} \frac{mg(z) + (z - z_0)g'(z)}{(z - z_0)g(z)} = \frac{mg(z_0) + (z_0 - z_0)g'(z_0)}{\frac{\partial}{\partial z}|_{z_0}(z - z_0)g(z)} \\
&= \frac{mg(z_0)}{g(z)|_{z=z_0} - (z - z_0)g'(z)|_{z=z_0}} = \frac{mg(z_0)}{g(z_0)} = m
\end{aligned}$$

□